MTH 161 - Fall 2013

Lecture 5

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LIMITS INVOLVING INFINITY

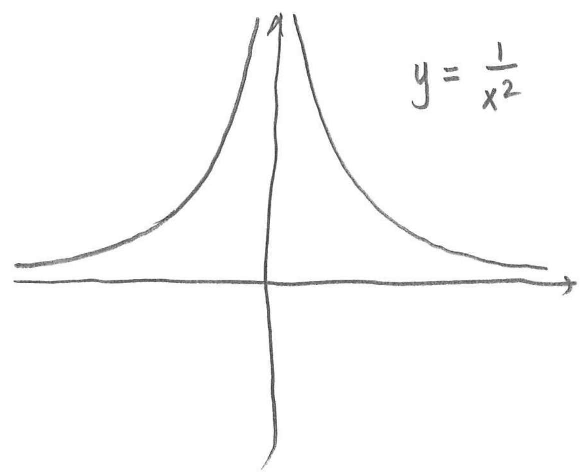
So what we would like to investigate the global behavior of functions, in particular whether their graphs approach asymptotes, vertical or horizontal.

INFINITE LIMITS

$$\lim_{x \to 0} \frac{1}{x^2}$$

Table

x	f(x)	x	f(x)
0.5	4	-0.5	4
0.2	25	-0.2	25
0.1	100	-0.1	100
0.05	400	-0.05	400
0.01	10,000	-0.01	10,000
0.001	1,000,000	-0.001	1,000,000



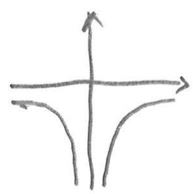
So we say $\lim_{x \to 0} \frac{1}{x^2} = \infty$

What is happening here is that the values of $f(x)$ becomes larger and larger as x-approaches a.

In this case we write $\lim_{x \to a} f(x) = \infty$

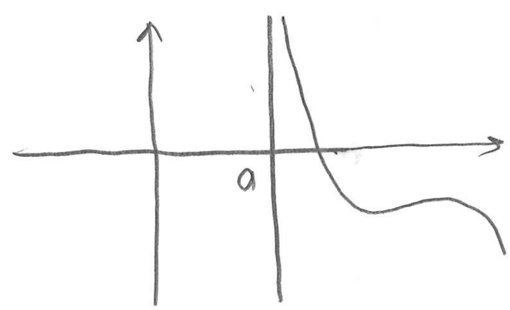
On the other hand if values of $f(x)$ becomes more and more negatives as x-approaches a. In this case we write $\lim_{x \to a} f(x) = -\infty$

$$\lim_{x \to 0} \left(-\frac{1}{x^2}\right) = -\infty$$

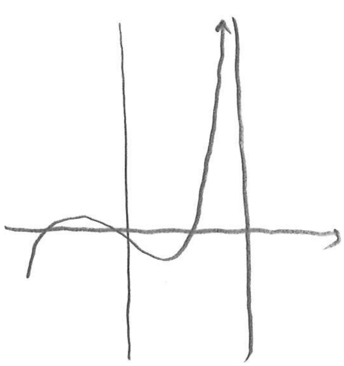


Similar definition can be given for one sided

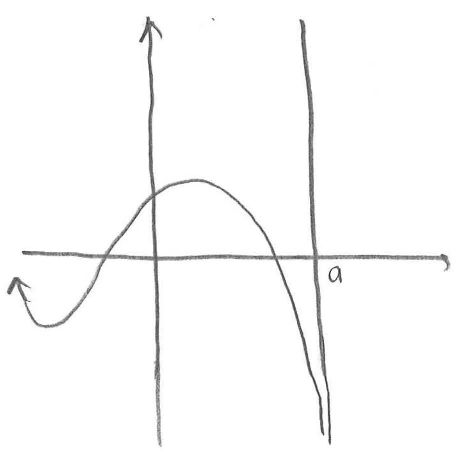
I will explain these with the help of graphs



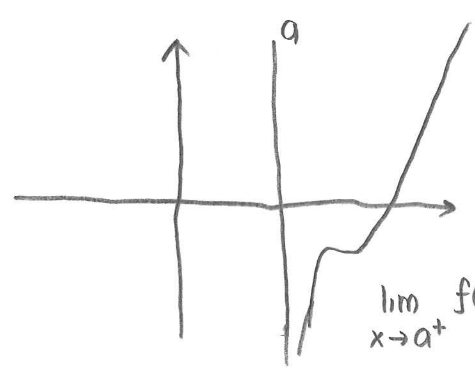
$\lim_{x \to a^+} f(x) = \infty$



$\lim_{x \to a^-} f(x) = \infty$



$\lim_{x \to a^-} f(x) = -\infty$



$\lim_{x \to a^+} f(x) = -\infty$

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DEFINITION

The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true :

$$\lim_{x \to a} f(x) = \infty \quad ; \quad \lim_{x \to a} f(x) = -\infty \quad ; \quad \lim_{x \to a^-} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \quad , \quad \lim_{x \to a^+} f(x) = \infty \quad , \quad \lim_{x \to a^+} f(x) = -\infty$$

Ex $\lim_{x \to 3} \frac{2x}{x-3}$

What happens when you plug in 3 , you get $\frac{6}{0}$

So in this case what you need to do is find left handed and right handed limits .

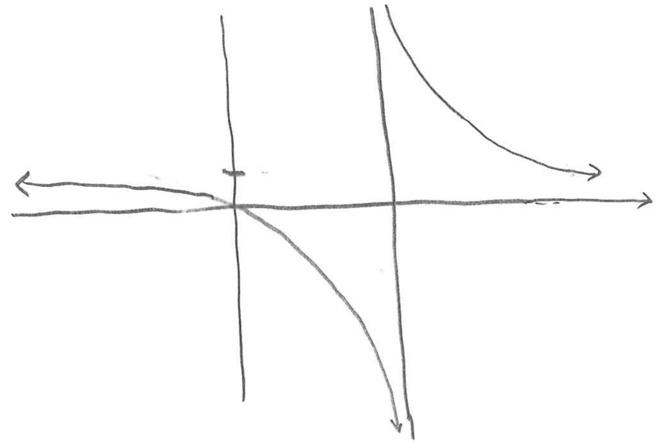
$$\lim_{x \to 3^+} \frac{2x}{x-3} = +\infty \text{ or } -\infty$$

So plug in 3.001 , $\frac{2(3.001)}{3.001 - 3} = \frac{6.002}{.001} > 0$.

Hence , $\lim_{x \to 3^+} \frac{2x}{x-3} = +\infty$

Similarly,

$$\lim_{x \to 3^-} \frac{2x}{x-3} = +\infty \text{ or } -\infty$$



Plug in 2.999

$$\frac{2(2.999)}{2.999-3} = \frac{5.998}{0.001} > 0$$

That means that $\displaystyle\lim_{x \to 3^-} \frac{2x}{x-3} = +\infty$

So, $\displaystyle\lim_{x \to 3} \frac{2x}{x-3}$ DNE.

Ex $$\lim_{x \to 3} \underbrace{\frac{(x-3)^2+6}{(x-3)^2}}_{}$$

$f(x) = \dfrac{(x-3)^2+6}{(x-3)^2}$

Plug in 3, we get $\frac{6}{0}$, so in this case we have

$$\lim_{x \to 3^+} f(x) = +\infty \quad \text{or} \quad \lim_{x \to 3^+} f(x) = -\infty$$

$\displaystyle\lim_{x \to 3^-} f(x) = +\infty$ or $-\infty$ and we need to determine which one it is.

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So what is $\lim_{x \to 3^+} f(x) = ?$

Plug in 3.001, $\dfrac{(3.001-3)^2+6}{(.001)^2} > 0$, hence $\lim_{x \to 3^+} f(x) = \infty$

$\lim_{x \to 3^-} f(x) = ?$

Plug in 2.999, $\dfrac{(2.999-3)^2+6}{(2.999-3)^2} > 0$, hence $\lim_{x \to 3^-} f(x) = \infty$

Hence, $\lim_{x \to 3} f(x) = \infty$

So the point is we can either use graphs or this idea.

LIMITS AT INFINITY

Earlier we considered cases, we let x approach a number and the result was that the values of y become arbitrarily large (positive or negative)

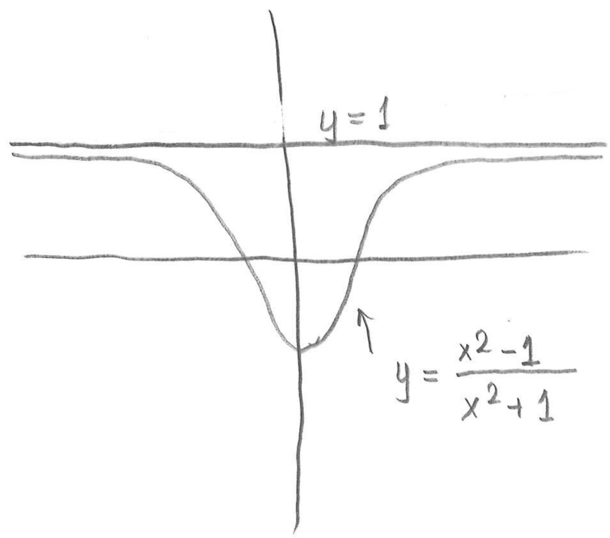
Now, we will let x becomes arbitrarily large (positive or negative) and see what happens to y.

Let us consider the behavior of the function f defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

let us make the table

x	$f(x)$
± 5	0.923077
± 10	0.980198
± 50	0.999200
± 100	0.999800
± 1000	0.999998



So as x grows larger and larger, values of $f(x)$ get closer and closer to 1.

So in this case we say $\lim_{x \to \infty} f(x) = L$.

<u>Defn</u> Let f be a function defined on some interval (a, ∞). Then

$\lim_{x \to \infty} f(x) = L$ means that the values of $f(x)$ can be made as close to L

as we like by taking x sufficiently large.

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Similar definition for $\lim_{x \to -\infty} f(x)$

So f be a function defined on some interval $(-\infty, a)$. Then $\lim_{x \to -\infty} f(x) = L$

means that values of $f(x)$ can be made as close to L as we like by taking

x sufficiently large negative.

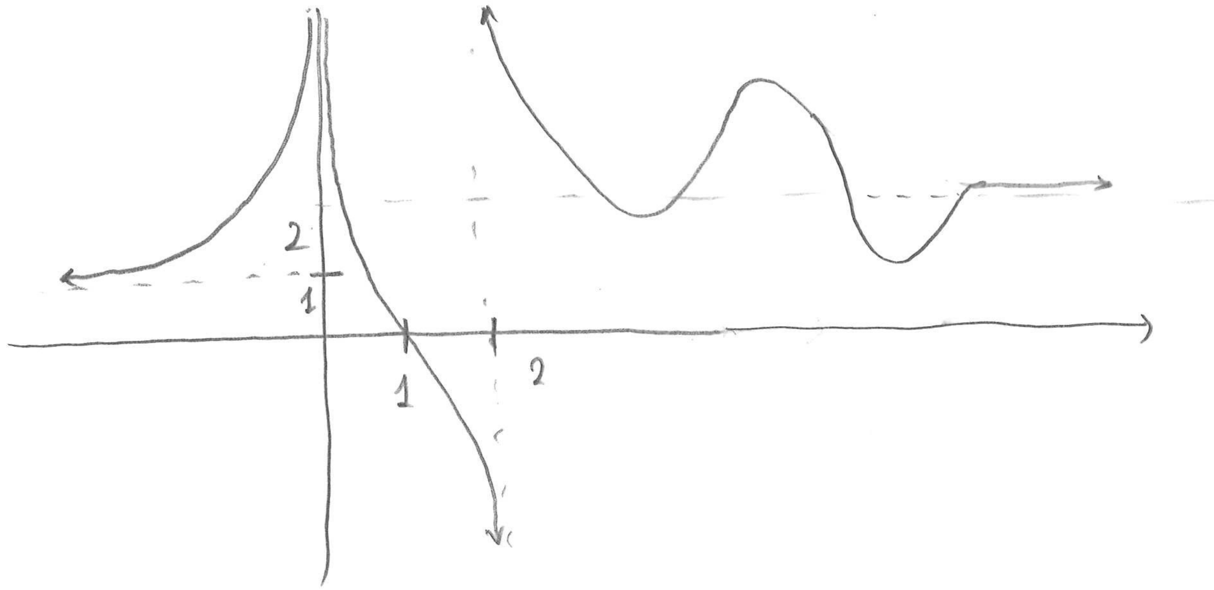
Let's go back to the earlier example

What is $\lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = ?$, 1.

Def^n The line $y = L$ is called a horizontal asymptote of the curve

$y = f(x)$ if either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$.

Ex Infinite limits, limits at infinity, asymptotes of the following function



So what is $\lim_{x \to 0^-} f(x) = \infty$, $\lim_{x \to 0^+} f(x) = \infty$, $\lim_{x \to 0} f(x) = \infty$

$\lim_{x \to 2^-} f(x) = -\infty$, $\lim_{x \to 2^+} f(x) = \infty$

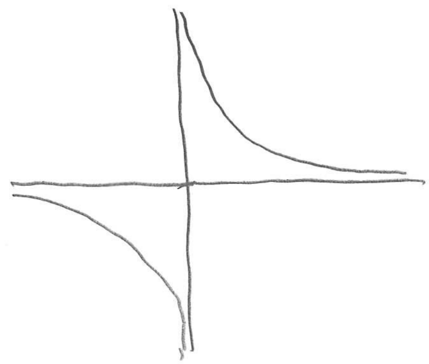
$\lim_{x \to \infty} f(x) = 2$, $\lim_{x \to -\infty} f(x) = 1$

So Vertical asymptotes at $x = 0$, $x = 2$

Horizontal " at $y = 1$, $y = 2$.

Imp $\lim_{x \to \infty} \frac{1}{x} = 0$, $\lim_{x \to -\infty} \frac{1}{x} = 0$

So horizontal asymptote at $y = 0$.



In general, for n a positive integer,

$\lim_{x \to \infty} \frac{1}{x^n} = 0$, $\lim_{x \to -\infty} \frac{1}{x^n} = 0$

Ex $\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$

Seems like as x gets larger, both numberator and denominator seem like it is increasing. So what about the ratio ? Need to do some algebra

To evaluate limits at infinity of any rational function, we divide both the numerator and denominator by the highest power in the denominator

(We may assume $x \neq 0$, because we are only interested in large values of x)

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2+4x+1} = \lim_{x\to\infty} \frac{\frac{3x^2-x-2}{x^2}}{\frac{5x^2+4x+1}{x^2}}$$

$$= \frac{\lim\limits_{x\to\infty}\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\lim\limits_{x\to\infty}\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)} = \frac{\lim\limits_{x\to\infty} 3 - \lim\limits_{x\to\infty}\frac{1}{x} - 2\lim\limits_{x\to\infty}\frac{1}{x^2}}{\lim\limits_{x\to\infty} 5 - 4\lim\limits_{x\to\infty}\frac{1}{x} + \lim\limits_{x\to\infty}\frac{1}{x^2}}$$

$$= \frac{3-0-0}{5-0-0} = \frac{3}{5}$$

A similar calculation show that lim as $x \longrightarrow -\infty$ is also $\frac{3}{5}$.

<u>Ex</u> Compute $\lim\limits_{x\to\infty} \left(\sqrt{x^2+1} - x\right)$

Since both $\sqrt{x^2+1}$ and x become really large as x becomes large, we can't really predict what happens to $\sqrt{x^2+1} - x$

$$\lim_{x\to\infty} \sqrt{x^2+1} - x \cdot \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} = \lim_{x\to\infty} \frac{(\sqrt{x^2+1})^2 - x^2}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty} \frac{x^2+1-x^2}{\sqrt{x^2+1}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} = 0.$$

Infinite limits at infinity

- $\lim_{x \to \infty} x^n = \infty$

- $\lim_{x \to -\infty} x^n = -\infty$ for n odd, $\lim_{x \to -\infty} x^n = \infty$ for n even

Ex $\lim_{x \to -\infty} \dfrac{x^4 + x^2 + 1}{x^2 + 3x + 2}$

Again divide by the highest power in denominator

$$\lim_{x \to -\infty} \frac{\dfrac{x^4 + x^2 + 1}{x^2}}{\dfrac{x^2 + 3x + 2}{x^2}} = \frac{\lim_{x \to -\infty} \left(x^2 + 1 + \frac{1}{x^2}\right)}{\lim_{x \to -\infty} \left(1 + \frac{3}{x} + \frac{1}{x^2}\right)}$$

$$= \frac{\lim_{x \to -\infty} x^2 + \lim_{x \to -\infty} 1 + \lim_{x \to -\infty} \frac{1}{x^2}}{\lim_{x \to -\infty} 1 + 3 \lim_{x \to -\infty} \frac{1}{x} + \lim_{x \to -\infty} \frac{1}{x^2}} = \frac{\lim_{x \to -\infty} x^2 + 1}{1} = \infty$$

Ex $\lim_{x \to \infty} x^2 - x = \underbrace{\infty - \infty}$

we don't know what this

However we can write

$\lim_{x \to \infty} x(x-1) = \infty$ since both x and x-1 becomes arbitrarily large.

Ex $\lim_{x \to -\infty} x^2 - x = +\infty + \infty = \infty$

both get arbitrarily large, so should their sum.